Easy lambda-terms are not always simple
- extended abstract -

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Lambda theories are congruences on the set of λ-terms, which contain β-conversion; they arise by either syntactical or semantical considerations. Indeed, a λ-theory may correspond to a possible operational semantics of λ-calculus, as well as it may be induced by a model of λ-calculus through the kernel congruence relation of the interpretation function. Lambda calculus has been originally investigated by using mainly syntactical methods (see Barendregt's book [7]). Syntactical proofs of consistency of remarkable λ-theories (for example, the theory equating all unsolvable λ-terms) were given in Barendregt's thesis [6]. Many other interesting examples of consistent λ-theories are studied in [7, Chapters 16,17], most of the time syntactically.

Since syntactic techniques are usually difficult to use in the study of λ-theories, then semantical methods have been extensively investigated. After the first model, found by Scott in 1969 in the category of complete lattices and Scott continuous functions, a large number of mathematical models for λ-calculus, arising from syntax-free constructions, have been introduced in various Cartesian closed categories (ccc, for short) of domains and were classified into semantics according to the nature of their representable functions, see e.g. [7, 12, 29]. Scott continuous semantics [31] is the class of reflexive cpo-models, that are reflexive objects in the category **Cpo** whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry [15]) and the strongly stable semantics (Bucciarelli-Ehrhard [16]) are refinements of the continuous semantics, introduced to approximate the notion of "sequential" Scott continuous function. Although Scott continuous semantics and the other mentioned semantics are structurally and equationally rich (each of them has $2^{\aleph_0}$ models inducing pairwise distinct λ-theories [26, 27]), nevertheless, they do not match all possible operational semantics of λ-calculus, because there is a continuum of λ-theories which are omitted by all ordered models of λ-calculus with a bottom element (see Honsell-Ronchi [20]; Salibra [30]).

Some of the models in the above semantics, called webbed models, are built from lower level structures called "webs". The simplest class of webbed models is the class of graph models, which was isolated in the seventies by Plotkin, Scott and Engeler [18, 29, 32] within the continuous semantics. The class of graph models contains the simplest models of λ-calculus, is itself the easiest describable class, and represents nevertheless a continuum of (non-extensional) λ-theories. Another example of a class of webbed models, and the most established one, is the class of filter models. It was isolated at the beginning of the eighties by Barendregt, Dezani and Coppo [8], after the introduction of the intersection type discipline by Coppo and Dezani [17]. Not all filter models live in Scott continuous semantics because, for example, some of them were introduced for the stable semantics (see Honsell-Ronchi [19]; Bastonero et al. [9]).

According to Jacopini [22] a closed λ-term M is *easy* if, for any other closed term N, the λ-theory generated by the equality $M = N$ is consistent. Easy terms can be considered computational processes of a completely non-informative kind. Thus they are suitable candidates for representing inside λ-calculus the undefined value of a partial recursive function. The paradigmatic unsolvable term $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ was shown easy by Jacopini [22] (cf. [7, p. 402]) with a syntactic proof. Other syntactical proofs that certain terms are easy may be found in the literature, e.g., (Jacopini-Venturini Zilli [24, 23]; Intrigila [21]; Berarducci-Intrigila [11]; Kuper [28]).

Baeten and Boerboom gave in [5] the first semantical proof of the easiness of Ω by showing that, for all closed terms M one can build a graph model satisfying the equation $\Omega = M$. Baeten and Boerboom build their graph model by a method of "forcing", which, although much simpler than the forcing techniques used in set theory, is somewhat in the same spirit. Forcing considerations have been extended by Zylberajch [33] to prove the simultaneous easiness of the members of some infinite family of easy terms (see also Berline-Salibra [14] and Berarducci [10]). However, the semantical methods via graph models have concrete limitations. For example, no semantical proof of the easiness of $\omega_3\omega_3\mathbf{I}$ (where $\omega_3 \equiv \lambda x.xxx$ and $\mathbf{I} \equiv \lambda x.x$) via graph models can exist, in contrast to the case Ω, since Kerth [25] has shown that no graph model satisfies the identity $\omega_3\omega_3\mathbf{I} = \mathbf{I}$. The easiness of the term $\omega_3\omega_3\mathbf{I}$ was proved syntactically in (Jacopini-Venturini Zilli [23]), but was only given a semantic proof in (Alessi et al. [2]), where the authors build, for each closed term M, a filter model of $\omega_3\omega_3\mathbf{I} = M$.

Alessi and Lusin in [4] introduced a general technique to prove the easiness of λ-terms through the notion of *simple easiness*. This notion implies easiness and can be handled in a natural way by semantic tools. It allows to prove consistency results via construction of suitable filter models of λ-calculus living in the category $\mathbf{Cpo}$: given a simple easy term M and an arbitrary closed term N, it is possible to build (in a canonical way) a non-trivial filter model which equates the interpretation of M and N. In [3] Alessi, Dezani and Lusin prove in such a way the easiness of several terms. Besides, simple easiness is interesting in itself, since it has to do with minimal sets of axioms which are needed in order to assign certain types to easy terms.

The TLCA list of open problems is a list of twenty-two problems that aims at collecting unresolved questions in the subject areas of the TLCA (Typed Lambda Calculi and Applications) series of conferences. Problem 1 and Problem 20 are the only ones that have been solved to date. Problem 19 in the TLCA list was posed by Fabio Alessi and Mariangiola Dezani-Ciancaglini in 2002 (see [1]) and asks whether easiness implies simple easiness. In this paper we negatively answer the question providing a nonempty co-r.e. (complement of a recursively enumerable) set of easy, but non simple easy, λ-terms.

Outline of the proof. The main idea is to apply computability theory in the context of lambda models, as done in [13]. The key step for the proof is the construction of a λ-model $\mathbf{P}$ with the following properties:

(i) $\mathrm{Ord}(\mathbf{P})$ is contained within $\mathrm{Ord}(\mathbf{F})$, for every filter model which lives in $\mathbf{Cpo}$ (being $\mathrm{Ord}(\mathbf{A}) = \{(M, N) : |M|^{\mathbf{A}} \leq_{\mathbf{A}} |N|^{\mathbf{A}}\}$);
(ii) the interpretation $|\lambda x.x|^{\mathbf{P}}$ is decidable.

We now briefly explain how such properties are obtained by our construction.

First of all we observe that for any filter model $\mathbf{F}$ in $\mathbf{Cpo}$ and any inequality $M \leq N$ which fails in $\mathbf{F}$, i.e., $|M|^{\mathbf{F}} \not\leq_{\mathbf{F}} |N|^{\mathbf{F}}$ there is a finite piece of $\mathbf{F}$ which is *responsible* for this failure. To such finite piece, let's say F_0, which is just a partial model of λ-calculus rather than an actual one, we apply a *completion procedure* whose outcome is a model $\mathbf{F}_\omega$ such that $M \leq N$ fails $\mathbf{F}_\omega$. Now $\mathbf{P}$ is defined as the direct product of all completions of finite pieces of filter models; as a direct product of λ-models, $\mathbf{P}$ itself is a λ-model and by construction every inequality which holds in $\mathbf{P}$ also holds in every filter model in $\mathbf{Cpo}$. This explains property (i).

The completion procedure that we use is also *effective* and each underlying set of the completion $\mathbf{F}_\omega$ of a finite piece F_0 of filter model admits a numeration, with respect to which the interpretation $|\lambda x.x|^{\mathbf{F}_\omega}$ is decidable. Then, by construction, $\mathbf{P}$ itself comes equipped with a numeration with respect to which the interpretation $|\lambda x.x|^{\mathbf{P}}$ is decidable. This roughly explains property (ii).

With these properties at hand we are now in the position of exhibiting a non-empty set of easy but non-simple easy terms.

By direct calculation, property (ii) implies that $X = \{N \in \Lambda^o : |N|^{\mathbf{P}} \leq |\lambda x.x|^{\mathbf{P}}\}$ is a non-empty beta-closed co-r.e. set (i.e. it is the complement of a recursively enumerable set) of λ-terms and moreover the set E of all easy terms is also beta-closed and co-r.e.; now a theorem of Visser allows us to say that $E \cap X$ is co-r.e. and non-empty too. Finally using property (i) we can prove that the assumption of simple easiness for an arbitrary term belonging to $E \cap X$ leads to the contradiction of Böhm's Theorem, so that set $E \cap X$ witnesses the existence of easy but non-simple easy terms.

References

1. F. Alessi, M. Dezani-Ciancaglini, TLCA list of open problems: Problem 19, http://tlca.di.unito.it/opltlca/problem19.pdf (2002).
2. F. Alessi, M. Dezani-Ciancaglini, F. Honsell, Filter models and easy terms, in: ICTCS '01: Proceedings of the 7th Italian Conference on Theoretical Computer Science, Springer-Verlag, London, UK, 2001.
3. F. Alessi, M. Dezani-Ciancaglini, S. Lusin, Intersection Types and Domain Operators, Theoret. Comput. Sci. 316 (1–3) (2004) 25–47.
4. F. Alessi, S. Lusin, Simple easy terms, Electronic Notes in Theoretical Computer Science 70 (1) (2003) 1 – 18, iTRS '02, Intersection Types and Related Systems (FLoC Satellite Event).
5. J. Baeten, B. Boerboom, Omega can be anything it should not be, in: Proc. Koninklijke Netherlandse Akademie van Wetenschappen, Serie A, Indag. Matematicae 41, 1979.
6. H. P. Barendregt, Some extensional term models for combinatory logics and λ-calculi, Master's thesis, University of Utrecht, dissertation (1971).
7. H. P. Barendregt, The Lambda calculus: Its syntax and semantics, North-Holland, Amsterdam, 1984.
8. H. P. Barendregt, M. Coppo, M. Dezani-Ciancaglini, A filter lambda model and the completeness of type assignment, Journal of Symbolic Logic 48 (4) (1983) 931–940.

9. O. Bastonero, A. Pravato, S. Ronchi Della Rocca, Structures for lazy semantics, in: Gries, de Roever (eds.), Programming Concepts and Methods (PROCOMET'98), Chaptman & Hall, 1998.
10. A. Berarducci, Infinite lambda-calculus and non-sensible models, in: Logic and Algebra (Pontignano 1994) (Lecture Notes in Pure and Applied Mathematics Series/180), Marcel Dekker Inc., 1996.
11. A. Berarducci, B. Intrigila, Some new results on easy lambda-terms, Theoretical Computer Science 121 (1-2) (1993) 71 – 88.
12. C. Berline, From computation to foundations via functions and application: The λ-calculus and its webbed models, TCS 249 (2000) 81–161.
13. C. Berline, G. Manzonetto, A. Salibra, Effective lambda models versus recursively enumerable lambda theories, Mathematical Structures in Computer Science 19 (2009) 897 – 942.
14. C. Berline, A. Salibra, Easiness in graph models, Theoretical Computer Science 354 (2006) 4–23.
15. G. Berry, Stable models of typed lambda-calculi, in: ICALP'78, LNCS 62, Springer-Verlag, Berlin, 1978.
16. A. Bucciarelli, T. Ehrhard, Sequentiality and strong stability, in: LICS'91, IEEE Computer Society Press, 1991.
17. M. Coppo, M. Dezani-Ciancaglini, An extension of the basic functionality theory for the λ-calculus, Notre-Dame Journal of Formal Logic 21 (4) (1980) 685–693.
18. E. Engeler, Algebras and combinators, Algebra Universalis 13 (3) (1981) 389–392.
19. F. Honsell, S. Ronchi Della Rocca, Reasoning about interpretations in qualitative lambda models, in: Proceedings of the IFIP Working Group 2.2/2.3, M. Broy and C.B. Jones editors, North-Holland, 1990.
20. F. Honsell, S. Ronchi Della Rocca, An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus, Journal of Computer and System Sciences 45 (1992) 49–75.
21. B. Intrigila, A problem on easy terms in lambda-calculus, Fundamenta Informaticae 15 (1) (1991) 99–106.
22. G. Jacopini, A condition for identifying two elements of whatever model of combinatory logic, in: Proceedings of the Symposium on Lambda-Calculus and Computer Science Theory, Springer-Verlag, London, UK, 1975.
23. G. Jacopini, M. Venturini-Zilli, Easy terms in the lambda-calculus, Fundamenta Informaticae 8 (2) (1985) 225–233.
24. G. Jacopini, M. V. Zilli, Equating for recurrent terms of λ-calculus and combinatory logic, Tech. Rep. 85 (1978).
25. R. Kerth, Isomorphism et équivalence équationnelle entre modèles du λ-calcul, Master's thesis, Université de Paris 7, thèse (1995).
26. R. Kerth, Isomorphism and equational equivalence of continuous lambda models, Studia Logica 61 (1998) 403–415.
27. R. Kerth, On the construction of stable models of λ-calculus, TCS 269 (2001) 23–46.
28. J. Kuper, On the jacopini technique, Information and Computation 138 (2) (1997) 101 – 123.
29. G. D. Plotkin, Set-theoretical and other elementary models of the λ-calculus, Theor. Comput. Sci. 121 (1-2) (1993) 351–409.
30. A. Salibra, Topological incompleteness and order incompleteness of the lambda calculus, ACM Transactions on Computational Logic (4) (2003) 379–401.
31. D. S. Scott, Continuous lattices, in: Toposes, algebraic geometry and logic, Springer-Verlag, Berlin, 1972.
32. D. S. Scott, Data types as lattices, SIAM J. Comput. (5) (1976) 522–587.
33. C. Zylberajch, Syntaxe et sémantique de la facilité en λ-calcul, Thèse de doctorat d'état (1991).